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ON DETERMINATION OF THE BASIC CHARACTERISTICS OF FREE CONVECTIVE HEAT EXCHANGE NEAR A FLAT VERTICAL SURFACE

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The results of mathematical modeling of free convective heat exchange near a semiinfinite, impermeable, flat vertical surface have been presented. The features of velocity and temperature fields as functions of the boundary conditions and the Prandtl number have been studied. Tables of numerical solutions have been given. The esults obtained have been compared to the numerical data of other authors.

Introduction. The significance of the theory of free convective heat exchange is growing steadily every year, which has been predetermined by the objective trends of modern technology toward developing with continuous increase in the unit powers of machines, mechanisms, and equipment, the complication and variety of their structures, and the difference in operating conditions and strategy with the simultaneous insufficiency of manpower and productive forces. The state of the art in this field of knowledge is characterized by the higher faithfulness of prediction of hydrodynamic and thermal processes, the development of new mathematical models and the refinement of old ones, the search for efficient numerical and analytical approaches to an analysis of applied problems formulated in the language of equations, and the extension of the fields of application of the data obtained [1]. Hundreds of names are counted at present in the bibliography on investigation of free convection on a vertical surface. The formulation and solution of problems within the framework of this issue even for the laminar regime of motion of a liquid involve certain methodological and mathematical difficulties. The interrelation of hydrodynamic and temperature fields, the nonlinear dependence of the basic characteristics on the governing parameters, the limited capabilities of analytical approaches, and the necessity of allowing, in numerical integration, for the fact that the solutions sought have both rapidly and slowly varying components at a time determine the belonging of these problems to a complex class of partial differential equations. The main drawback of numerous results is that one uses, in them, data from the earlier works in which no comprehensive information of a numerical analysis of the problem is given. Therefore, comparisons are usually made in a narrow parametric range, which, naturally, makes it difficult to clearly visualize some constructed solutions or others and the degree to which the entire problem has been attacked. This information base often provides the basis for analysis of other problems, e.g., on calculation of conjugate convective heat exchange or study of nonstationary heat transfer.

Thus, the need for additional more detailed investigations of free convective heat exchange is apparent.

Below, we give results of a comprehensive numerical analysis of fully developed laminar free convective flows on a semiinfinite, impermeable, flat vertical plate with two types of thermal boundary conditions: a constant wall temperature and a constant heat flux on the surface.

Basic Equations. Using the assumptions made in boundary-layer theory, in combination with the Boussinesq approximation, we may write the basic equations for the problem in question in the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta\,(T - T_\infty)\,, \quad u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\nu}{\mathrm{Pr}}\frac{\partial^2 \grave{O}}{\partial y^2}\,. \tag{1}$$

Boundary conditions for system (1) will be

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TABLE 1. Comparison of the Values of $f''(0, \mathrm{Pr})$ for the Case of an Isothermal Surface

Pr	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]
0.01	1.3947	—	1.395266	—	—	—	—	—	1.39625	1.3969
0.02	—	1.35623	—	—	—	—	—	—	—	1.3562
0.03	—	1.32710	—	—	—	—	—	—	—	—
0.05	—	—	—	—	—	—	—	—	—	1.2844
0.10	—	—	—	—	1.21501	—	—	1.2140	1.21495	1.2150
0.50	—	—	—	—	—	—	—	—	—	1.0086
0.70	—	—	—	—	0.96012	—	—	—	—	0.9601
0.72	—	—	0.955977	—	—	—	—	0.9558	0.95601	—
0.733	0.9533	—	—	—	—	—	0.95343	—	—	—
1	0.9081	—	0.908178	—	—	—	—	0.9081	0.90821	0.9082
2	0.8079	—	—	0.80788	—	—	—	—	0.80794	0.8079
5	—	—	0.681348	—	—	—	—	—	0.68137	0.6814
6.7	—	—	—	—	—	—	—	—	0.64318	—
7	—	—	—	—	—	—	—	—	—	0.6375
10	0.5928	—	0.592806	0.59284	—	0.59284	0.59284	0.5930	0.59284	0.5928
100	0.3560	—	—	—	—	0.35594	0.35594	0.3564	0.35596	0.3559

TABLE 2. Comparison of the Values of $-h'(0, \mathrm{Pr})$ for the Case of an Isothermal Surface

Pr	[2]	[3]	[4]	[5]	[12]	[6]	[7]	[8]	[9]	[10]	[11]
0.01	0.0574	—	0.057365	—	—	—	—	—	—	0.0571	0.0570
0.02	—	0.0789	—	—	—	—	—	—	—	—	0.0789
0.03	—	0.0952	—	—	—	—	—	—	—	—	—
0.05	—	—	—	—	—	—	—	—	—	—	0.1200
0.10	—	—	—	—	—	0.1629	—	—	0.1637	0.1627	0.1627
0.50	—	—	—	—	—	—	—	—	—	—	0.3120
0.70	—	—	—	—	—	0.3532	—	—	—	—	0.3532
0.72	0.3568	—	0.356819	—	—	—	—	—	0.3567	0.3568	—
0.733	0.3592	—	—	—	—	—	—	0.35915	—	—	—
1	0.4010	—	0.401026	—	0.4010	—	—	—	0.4009	0.4010	0.4010
2	0.5066	—	—	0.50662	—	—	—	—	—	0.5066	0.5066
5	—	—	0.674576	—	—	—	—	—	0.6750	—	0.6746
6.7	—	—	—	—	—	—	—	—	0.7360	—	—
7	—	—	—	—	—	—	—	—	—	—	0.7455
10	0.8269	—	0.826807	0.82684	—	—	0.82684	0.82659	0.8266	0.8268	0.8268
100	1.5490	—	—	—	1.5493	—	1.54953	1.54953	1.5495	1.5495	1.5495

$$y = 0: \quad v = u = 0\,, \quad T = T_{\mathrm{w}} \quad \text{or} \quad -k\frac{\partial T}{\partial y} = q_{\mathrm{w}}\,;$$

$$y \to \infty: \quad u \to 0\,, \quad T \to T_{\infty}\,. \tag{2}$$

The next step is the reduction of Eqs. (1) and (2) to dimensionless form. This procedure is more conveniently performed individually for the cases of a constant wall temperature and a constant heat flux on the surface.

Isothermal Surface. We determine the dimensionless temperature

TABLE 3. Characteristics of Free Convective Heat Exchange on a Semiinfinite, Impermeable, Isothermal Flat Vertical Surface

Pr	$f''(0, \mathrm{Pr})$	$-h'(0, \mathrm{Pr})$	$f'_m(\mathrm{Pr})$	$\eta(f'_m)$	$f(\infty, \mathrm{Pr})$
0.70	0.9601241	0.3532078	0.556854	1.3643	1.7143640
0.72	0.9560360	0.3568303	0.552487	1.3598	1.6938851
1	0.9081912	0.4010331	0.502656	1.3079	1.4792094
5	0.6813538	0.6745825	0.296961	1.0752	0.8574515
6.7	0.6431175	0.7359703	0.267089	1.0361	0.7881081
7	0.6374998	0.7455123	0.262816	1.0303	0.7783997
10	0.5928330	0.8268440	0.229905	0.9839	0.7049284
100	0.3559477	1.5495357	0.088425	0.7070	0.3864672
500	0.2426154	2.3498926	0.042355	0.5482	0.2573906

$$\eta = \left(\frac{g\beta\Delta T_{\mathrm{w}}}{\nu^2}\right)^{1/4} x^{-1/4} y \tag{3}$$

and introduce the dimensionless stream function and temperature

$$\psi(x, \eta) = (g\beta\Delta T_{\mathrm{w}}\nu^2)^{1/4} f(\eta)\, x^{3/4}, \quad \Delta T(x, \eta) = \Delta T_{\mathrm{w}} h(\eta). \tag{4}$$

As a result of substitution of relations (3) and (4) into (1) and (2), we obtain

$$f''' + \frac{3}{4} ff'' - \frac{1}{2} f'^2 + h = 0, \quad \frac{1}{\mathrm{Pr}} h'' + \frac{3}{4} fh' = 0, \quad f(0) = 0, \quad f'(0) = 0,$$
$$f'(\infty) = 0, \quad h(0) = 1, \quad h(\infty) = 0, \tag{5}$$

where $'$ means the derivative with respect to η.

The distribution of the longitudinal velocity component is described by the function $f' = u/(g\beta\Delta T_{\mathrm{w}} x)^{1/2}$, whereas the local Nusselt and Grashof numbers, the friction stress on the wall, and and mass rate of flow have the form

$$\mathrm{Nu}_x = \{-h'(0)\}\, \mathrm{Gr}_x^{1/4}, \quad \mathrm{Gr}_x = \frac{g\beta\Delta T_{\mathrm{w}} x^3}{\nu^2}, \quad \frac{\tau_{\mathrm{w}} x^2}{\rho\nu^2} = f''(0)\, \mathrm{Gr}_x^{3/4}, \quad \frac{m}{\mu} = f(\infty)\, \mathrm{Gr}_x^{1/4}. \tag{6}$$

Thus, determination of the characteristics of free convective heat exchange near a flat isothermal vertical surface has been reduced to integration of the system of two interrelated nonlinear ordinary differential equations with parameter Pr. Since no analytical solution of problem (5) was found, primary emphasis was placed on its numerical solution in specifying the Prandtl number within the framework of different computational procedures and algorithms [2–12] (Tables 1 and 2). It should be noted that the dependence of the functions sought on Pr is different:

$$\mathrm{Pr} \to \infty: \ f \sim \mathrm{Pr}^{-0.25}, \ h' \sim \mathrm{Pr}^{0.25}, \ f'' \sim \mathrm{Pr}^{-0.25};$$
$$\mathrm{Pr} \to 0: \ f \sim \mathrm{Pr}^{-0.5}, \ h' \sim \mathrm{Pr}^{0.5}, \ f'' \sim \mathrm{const}. \tag{7}$$

Therefore, if we seek to numerically solve Eqs. (5) without using special techniques for high or low values of the Prandtl number, this will cause large errors. By virtue of what has been stated above, it becomes expedient to change to another system ($'$ means the derivative with respect to ξ)

TABLE 4. Comparison of the Values of $f''(0, \mathrm{Pr})$ for the Case of a Constant Heat Flux on the Wall

Pr	[13]	[14]	[15]	[16]	[7]	[8]	[17]	[10]
0.01	—	—	—	—	—	—	—	119.7608
0.10	3.12850	—	—	—	—	—	—	3.1288
0.70	—	1.566044	—	—	—	—	—	—
0.72	—	—	—	—	—	—	—	1.5495
0.733	—	—	—	1.539925	—	1.53993	1.539925	—
1	1.37436	1.374382	1.37438	—	—	—	—	1.3743
2	—	1.064575	—	—	—	—	—	1.0645
3	—	0.915973	—	—	—	—	—	—
4	—	0.822914	—	—	—	—	—	—
5	—	0.757068	—	—	0.75706	—	—	0.7570
6.7	—	—	—	0.678333	—	—	0.678333	0.6782
7	—	0.667248	—	—	—	—	—	—
10	0.58326	0.583201	—	—	0.58320	0.58320	—	0.5833
50	—	—	—	—	0.31464	—	—	—
100	0.24024	—	—	—	0.24021	0.24021	—	0.2402
500	—	—	—	—	0.12751	—	—	—

$$F''' + \frac{1}{\mathrm{Pr}}\left(\frac{3}{4}FF'' - \frac{1}{2}F'^2\right) + \frac{1+\mathrm{Pr}}{\mathrm{Pr}}H = 0\,, \quad H'' + \frac{3}{4}FH' = 0\,,$$
$$F(0) = 0\,, \quad F'(0) = 0\,, \quad F'(\infty) = 0\,, \quad H(0) = 1\,, \quad H(\infty) = 0 \tag{8}$$

for solution of the initial problem. Relations (8) are a result of using new generalized variables:

$$f(\eta) = \mathrm{Pr}^{-1/2}(1+\mathrm{Pr})^{-1/4}F(\xi)\,, \quad \xi = \mathrm{Pr}^{1/2}(1+\mathrm{Pr})^{-1/4}\eta\,, \quad h(\eta) = H(\xi)\,. \tag{9}$$

Once Eqs. (5) have been reduced to the form (8), the parameters of flow are "normalized" so that their values vary within finite limits, e.g., $-H'(0)$ ranges between 0.47 and 0.6. The latter enables us to reduce problem (8) to the Cauchy problem with unknown parameters and to use the targeting method in finite form. Numerical integration has been carried out within the standard Runge–Kutta scheme. The data of the numerical solution are presented in Table 3.

Constant Heat Flux on the Surface. In this case, we have

$$\psi(x,\eta) = \left(\frac{g\beta\nu^3 q_w}{k}\right)^{1/5} f(\eta)\, x^{4/5}\,, \quad \eta = \left(\frac{g\beta q_w}{k\nu^2}\right)^{1/5} x^{-1/5} y\,, \quad u(x,\eta) = \left(\frac{g\beta q_w \nu^{1/2}}{k}\right)^{2/5} f'(\eta)\, x^{3/5}\,,$$
$$\Delta T(x,\eta) = \left(\frac{q_w \nu^{1/2}}{k\,(g\beta)^{1/4}}\right)^{4/5} h(\eta)\, x^{1/5}\,. \tag{10}$$

Substitution of (10) into (1) and (2) yields the following system of equations:

$$f''' + \frac{4}{5}ff'' - \frac{3}{5}f'^2 + h = 0\,, \quad \frac{1}{\mathrm{Pr}}h'' + \frac{4}{5}fh' - \frac{1}{5}f'h = 0\,, \quad f(0) = 0\,,$$
$$f'(0) = 0\,, \quad f'(\infty) = 0\,, \quad h'(0) = -1\,, \quad h(\infty) = 0\,. \tag{11}$$

TABLE 5. Comparison of the Values of $h(0, \mathrm{Pr})$ for the Case of a Constant Heat Flux on the Wall

Pr	[13]	[18]	[14]	[12]	[19]	[20]	[16]	[7]	[8]	[17]	[10]	[21]	[15]
0.001	—	—	—	—	21.1872	—	—	—	—	—	—	—	—
0.01	—	—	—	—	8.69544	—	—	—	—	—	8.6963	—	—
0.1	3.7952	—	—	—	3.79770	—	—	—	—	—	3.7961	3.7965	—
0.5	—	—	—	—	—	2.27955	—	—	—	—	—	—	—
0.7	—	—	2.068534	—	—	—	—	—	—	—	—	2.0687	—
0.72	—	—	—	2.0515	—	—	—	—	—	—	2.0526	—	—
0.733	—	—	—	—	—	2.04174	2.041734	—	2.04173	2.041734	—	—	—
1	1.8728	—	1.872839	1.8728	1.87280	1.87283	—	—	—	—	1.8733	—	1.87284
2	—	—	1.559111	—	—	1.55911	—	—	—	—	1.5593	—	—
3	—	—	1.408019	—	—	—	—	—	—	—	—	—	—
4	—	—	1.312465	—	—	—	—	—	—	—	—	—	—
5	—	—	1.244133	1.2431	—	—	—	1.24413	—	—	1.2445	—	—
6.7	—	—	—	—	—	—	1.161321	—	—	1.161320	1.1617	—	—
7	—	—	1.149541	—	—	—	—	—	—	—	—	1.1498	—
10	1.05889	—	1.059004	1.0589	1.05887	—	—	1.05900	1.05900	—	1.0590	—	—
50	—	—	—	—	—	—	—	—	—	—	—	—	—
100	0.64248	0.64251	—	0.6424	0.64249	—	—	0.64252	0.64252		0.6425	0.6433	—
500	—	—	—	—	—	—	—	—	—	—	—	—	—
1000	—	—	—	—	0.40014	—	—	—	—	—	—	—	—

TABLE 6. Characteristics of Free Convective Heat Exchange on a Semiinfinite, Impermeable, Flat Vertical Surface with a Constant Heat Flux

Pr	$f''(0, \mathrm{Pr})$	$h'(0, \mathrm{Pr})$	$f'_{\mathrm{m}}(\mathrm{Pr})$	$\eta\,(f'_{\mathrm{m}})$	$f(\infty, \mathrm{Pr})$
0.70	1.5660422	2.0685331	0.730285	1.1151	1.8820244
0.72	1.5500202	2.0520877	0.721625	1.1136	1.8556490
1	1.3743817	1.8728379	0.626716	1.0953	1.5821137
5	0.7570680	1.2441325	0.300576	0.9963	0.8257830
6.7	0.6783334	1.1613205	0.261020	0.9768	0.7459188
7	0.6672466	1.1495394	0.255515	0.9738	0.7348384
10	0.5832012	1.0590043	0.214384	0.9486	0.6518663
100	0.2402073	0.6425167	0.064047	0.7781	0.3152117
500	0.1275070	0.4608027	0.025964	0.6545	0.1931532

Among the physical quantities of interest are not only the velocity (temperature) distributions but also the local Nusselt and Grashof numbers, the friction stress on the wall, and the mass rate of flow. It may be shown that

$$\mathrm{Nu}_x = \frac{1}{h\,(0)}\,\mathrm{Gr}_x^{1/5}\,, \quad \frac{\tau_{\mathrm{w}} x^2}{\rho \nu^2} = f''\,(0)\,\mathrm{Gr}_x^{3/5}\,, \quad \frac{m}{\mu} = f\,(\infty)\,\mathrm{Gr}_x^{1/5}\,, \quad \mathrm{Gr}_x = \frac{g \beta q_{\mathrm{w}} x^4}{k \nu^2}\,. \tag{12}$$

Formulas (12) enable us to promptly determine and evaluate the action of different parameters and factors on the hydrodynamics and intensity of heat transfer. Numerical methods of analysis of the formulated problem (11) have been developed in a number of works [7, 8, 10, 12–21]. A comparison of the results obtained within the framework of different numerical schemes (Tables 4 and 5) yields one feature, namely, spread in calculated values. To overcome the difficulties of numerical integration we introduce new variables and functions:

$$f\,(\eta) = \mathrm{Pr}^{-3/5}\,(1 + \mathrm{Pr})^{-1/5}\,F\,(\xi)\,, \quad \xi = \mathrm{Pr}^{2/5}\,(1 + \mathrm{Pr})^{-1/5}\,\eta\,, \quad h\,(\eta) = \mathrm{Pr}^{-2/5}\,(1 + \mathrm{Pr})^{1/5}\,H\,(\xi)\,. \tag{13}$$

With account for (13), system (11) takes the form

$$F''' + \frac{1}{\Pr}\left(\frac{4}{5}FF'' - \frac{3}{5}F'^2\right) + \frac{1+\Pr}{\Pr}H = 0\,,\quad H'' + \frac{4}{5}FH' - \frac{1}{5}F'H = 0\,,\quad F(0) = 0\,,$$

$$F'(0) = 0\,,\quad F'(\infty) = 0\,,\quad H'(0) = -1\,,\quad H(\infty) = 0\,. \tag{14}$$

Problem (14) is devoid of the drawbacks of the system of equations (11) and resolves many difficulties arising in numerical analysis for high or low values of Pr numbers. The solution of (14) has been found with the Runge–Kutta integration scheme in combination with the Newton–Raphson method to satisfy the conditions at the external boundary of the boundary layer. The calculation results are presented in Table 6.

Calculation Results and Their Discussion. The results obtained enable us to elucidate the manner in which the Prandtl number influences the dynamic and thermal characteristics of free convective motion of a liquid along a vertical plate. It follows from Tables 3 and 6 that the calculated parameters, other than the quantity $-h'(0, \Pr)$, monotonically decrease with increase in Pr throughout the considered range of variation in Pr numbers. It is of interest to directly compare data found for the liquid with Prandtl numbers Pr = 0.7 and 7 that are characteristic of air and water. For any specified value of the parameter Gr_x, the local Nusselt number in the case T_w = const and q_w = const is higher for Pr = 7 than that for Pr = 0.7. The reason is that the thickness of the thermal boundary layer decreases with increase in the Prandtl number and the temperature gradient on the plate surface grows, which leads to a higher intensity of heat transfer. On the other hand, the friction factor for Pr = 0.7 is higher than that for Pr = 7. This result is attributed to the fact that liquids having a Prandtl number of 0.7 are characterized by larger variations in the velocity gradients on the wall than liquids with Pr = 7. A comparison of local Nusselt numbers for the cases q_w = const and T_w = const can be made based on the formula

$$\frac{(\mathrm{Nu}_x)_{q_\mathrm{w}=\mathrm{const}}}{(\mathrm{Nu}_x)_{T_\mathrm{w}=\mathrm{const}}} = \frac{1}{(h(0,\Pr))^{5/4}\,(-h'(0,\Pr))}\,. \tag{15}$$

Relation (15) is more than unity, i.e., heating at a constant density of the heat flux on the wall yields higher values of the local Nusselt number than heating at a constant wall temperature. This is due to the fact that liquid particles near the plate surface have a higher velocity and will remove heat in the case q_w = const more rapidly than at T_w = const. Furthermore, the ratio of the Nusselt numbers decreases with increase in Pr and becomes virtually constant and equal to 1.12 for high values of the Prandtl numbers.

NOTATION

g, free-fall acceleration, m/sec^2; Gr_x and Nu_x, local Grashof and Nusselt numbers; k, thermal conductivity, W/(m·K); m, mass rate of flow, kg/(m·sec); Pr, Prandtl number; q_w, heat flux on the wall, W/m^2; T, temperature, K; T_w and T_∞, wall and ambient temperature, K; u and v, longitudinal and transverse velocity components, m/sec; x, y, longitudinal and transverse coordinates, m; β, coefficient of volumetric thermal expansion, 1/K; $\Delta T = T - T_\infty$, excess temperature, K; μ, coefficient of dynamic viscosity, kg/(m·sec); ν, coefficient of kinematic viscosity, m^2/sec; ρ, density, kg/m^3; τ_w, friction stress on the wall, kg/(m·sec^2). Subscripts: m, maximum values; w, wall; ∞, ambient liquid.

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